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Derivation of monopole solutions by Hirota's method

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Abstract. The second-order field equations in the 't Hooft-Polyakov monopole theory in the Prasad-Sommerfield limit are solved by Hirota's method. All the known point and regular solutions are rederived in a systematic way.

1. Introduction

In recent times there has been considerable progress in our understanding of non-linear phenomena. In spite of this there exist relatively few systematic methods for obtaining solutions to non-linear differential equations. One such method, developed in the last decade, is the bilinear operator method of Hirota (1976, 1980). This method, capable of generating particular solutions, has been successfully applied to a number of time evolution equations of mathematical physics, especially those admitting soliton solutions. The method has also been used in the case of coupled non-linear equations (Hirota and Satsuma 1981). In this paper we apply Hirota's method to the coupled second-order field equations of the 't Hooft-Polyakov monopole theory ('t Hooft 1974, Polyakov 1974).

In 1974 't Hooft (1974) and Polyakov (1974) showed that magnetic monopoles can exist as finite-energy solutions of the field equations of non-Abelian gauge theories. Since the field equations are highly non-linear it has not been possible so far to obtain the solution in a closed form. However, in the limit of vanishing Higgs self-interaction, Prasad and Sommerfield (1975, hereafter referred to as PS) obtained an exact solution by guesswork. Later it was realised that this solution satisfies the lower bound of energy, called the Bogomolny bound (Bogomolny 1976), and hence could be obtained by solving first-order equations instead of second-order field equations. Several point monopole solutions were also obtained (Protogenov 1977, Ju 1978) by solving the first-order Bogomolny equations. It may be noted that even though all the solutions of the Bogomolny equations are solutions of the field equations the converse is not true.

Here we solve the second-order field equations of the 't Hooft-Polyakov monopole theory in the PS limit by applying Hirota's method. The solutions are obtained in the form of a power series involving a large number of arbitrary constants. Only for specific choices of these constants are we able to sum these series and obtain the solutions in terms of elementary functions. It happens, however, that where we have been able to put the solutions into closed form they satisfy the first-order Bogomolny equations and coincide with solutions already reported in the literature. Still it is interesting that all these monopole solutions could be obtained from the second-order field equations in a single framework and in a systematic way. Moreover, our work suggests the possible existence of additional monopole solutions, the demonstration of which is not complete at the moment because of the difficulty of summing certain series expansions.

In § 2 we apply Hirota's method to the field equations and obtain a general solution in terms of infinite series which depend on a number of parameters. By adjusting the parameters, the series are summed and different monopole solutions obtained. This is discussed in § 3. Some of the details of the calculations are relegated to an appendix.

2. Hirota's method for the 't Hooft–Polyakov equations

In the PS limit the equations of motion of the SU(2) gauge theory become (Prasad and Sommerfield 1975)

$$r^2\frac{\mathrm{d}^2K}{\mathrm{d}r^2}=K(K^2-1+H^2) \tag{1a}$$

$$r^2\frac{\mathrm{d}^2H}{\mathrm{d}r^2}=2HK^2 \tag{1b}$$

where r is the radial variable and K and H are functions of r only. The gauge and Higgs fields are non-singular at the origin only if

$$K\to 1 \qquad H\to 0 \qquad \text{as } r\to 0. \tag{2}$$

Boundary condition (2) is also necessary for finite energy of the solution.

Following Hirota we make a dependent variable transformation

$$K(r)=\frac{A(r)}{B(r)} \qquad H(r)=\frac{C(r)}{B(r)} \tag{3}$$

in terms of which (1) can be rewritten as

$$r^2(BD^2AB-AD^2BB)=A(A^2-B^2+C^2) \tag{4a}$$

$$r^2(BD^2CB-CD^2BB)=2CA^2 \tag{4b}$$

where the bilinear operator D^n is defined by

$$D^nAB=\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\mathrm{d}}{\mathrm{d}r'}\right)^nA(r)B(r')|_{r=r'}. \tag{5}$$

We split (4b) using an arbitrary function $f(r)$ to get

$$r^2D^2BB+fB^2=-2A^2 \tag{6a}$$

$$r^2D^2CB+fCB=0. \tag{6b}$$

One readily verifies that solutions of (6) above are solutions of (4b). Equation (4a) now becomes

$$r^2BD^2AB=A[C^2-(f+1)B^2-A^2]. \tag{6c}$$

In Hirota's method the dependent variables A, B and C are expanded in a perturbation series. A consistent expansion is

$$A(r)=\varepsilon A_1(r)+\varepsilon^2A_2(r)+\dots \tag{7a}$$

$$B(r)=1+\varepsilon B_1(r)+\varepsilon^2B_2(r)+\dots \tag{7b}$$

$$C(r)=1+\varepsilon C_1(r)+\varepsilon^2C_2(r)+\dots \tag{7c}$$

where ε is a parameter. Substituting (7) in (6) and comparing the zeroth power of ε on both sides we see that $f(r)$ should be equal to zero for consistency. Hence (6) can be rewritten as

$$r^2D^2BB = -2A^2 \tag{8a}$$

$$D^2CB = 0 \tag{8b}$$

$$r^2BD^2AB = A(C^2 - B^2 - A^2). \tag{8c}$$

The functions (A_1, B_1, C_1), (A_2, B_2, C_2), ..., are obtained by integrating successively the linear equations which follow by substituting (7) in (8) and comparing coefficients of $\varepsilon, \varepsilon^2, \ldots$, respectively. For example, the first two sets of equations are

$$\frac{d^2B_1}{dr^2} = 0 \qquad \frac{d^2C_1}{dr^2} = 0 \qquad \frac{d^2A_1}{dr^2} = 0 \tag{9a}$$

$$\begin{aligned} &r^2(2D^2B_21 + D^2B_1B_1) = -2A_1^2 \\ &D^2C_21 + D^2C_1B_1 + D^21B_2 = 0 \\ &r^2(D^2A_21 + D^2A_1B_1 + B_1D^2A_11) = 2A_1(C_1 - B_1). \end{aligned} \tag{9b}$$

The general solutions of (9a) are

$$B_1 = b_1r + d_1 \qquad C_1 = c_1r + e_1 \qquad A_1 = a_1r + f_1 \tag{10a}$$

where $a_1, \ldots, f_1$ are arbitrary constants. Substituting (10a) we integrate (9b) to obtain

$$\begin{aligned} &B_2 = (b_1^2 - a_1^2)(r^2/2!) - 2a_1f_1(r \ln r - r) + f_1^2 \ln r + b_2r + d_2 \\ &C_2 = (a_1^2 - b_1^2 + 2b_1c_1)(r^2/2!) + 2a_1f_1(r \ln r - r) - f_1^2 \ln r + c_2r + e_2 \\ &A_2 = a_1c_1r^2 + 2[f_1(c_1 - b_1) + a_1(e_1 - d_1)](r \ln r - r) - 2f_1(e_1 - d_1) \ln r + a_2r + f_2. \end{aligned} \tag{10b}$$

Successive terms are evaluated using the remaining equations to obtain A, B and C in the form of infinite series. These series may be summed by choosing the constants of integration suitably and the cases where this can be done to yield the solutions in terms of elementary functions are described in the next section.

3. Solutions

Choosing all integration constants to be zero we have

$$A_n = B_n = C_n = 0 \qquad \text{for } n = 1, 2, 3, \ldots \tag{11a}$$

so that

$$K = 0 \qquad H = 1. \tag{11b}$$

The choice $a_1 = b_1$, $d_1 = e_1$ with all other constants zero yields

$$B = 1 + \varepsilon(b_1r + d_1) \qquad C = 1 + \varepsilon d_1 \qquad A = \varepsilon a_1r \tag{12a}$$

and

$$K = \frac{\eta r}{1 + \eta r} \qquad H = \frac{1}{1 + \eta r} \tag{12b}$$

where $\eta = a_1\varepsilon/(1 + d_1\varepsilon)$ is an arbitrary constant.

For $a_1 \neq b_1$, $d_1 = e_1$ being the only non-zero constants, we find

$$A(r) = a_1 r \tag{13a}$$

$$B = 1 + d_1 + b_1 r + E^2(1 - d_1\varepsilon + d_1^2\varepsilon^2 - \ldots)\frac{\varepsilon^2 r^2}{2!}$$
$$+ b_1 E^2(1 - 2d_1\varepsilon + 3d_1^2\varepsilon^2 + \ldots)\frac{\varepsilon^3 r^3}{3!} + E^4(1 - 3d_1\varepsilon + 6d_1^2\varepsilon^2 - \ldots)\frac{\varepsilon^4 r^4}{4!}$$
$$+ b_1 E^4(1 - 4d_1\varepsilon + 10d_1^2\varepsilon^2 - \ldots)\frac{\varepsilon^5 r^5}{5!} + \ldots. \tag{13b}$$

Assuming $|d_1\varepsilon| < 1$, by binomial theorem, we have

$$B = 1 + \varepsilon d_1 + \varepsilon b_1 r + \frac{E}{1+d_1}\frac{(\varepsilon r)^2}{2!} + \frac{bE^2}{(1+d_1)^2}\frac{(\varepsilon r)^3}{3!} + \ldots$$
$$= (\varepsilon/\eta)(b_1 \sinh \eta r + E \cosh \eta r) \tag{14}$$

where

$$E^2 = b_1^2 - a_1^2 \qquad \text{and} \qquad \eta = \frac{E\varepsilon}{1 + d\varepsilon}. \tag{15}$$

Due to the arbitrariness of d and ε we can take η as an arbitrary constant independent of a_1 and b_1. After a straightforward similar calculation the series for $C(r)$ becomes

$$C(r) = (\varepsilon/\eta)\{(E - b\eta r)\cosh \eta r + (b - E\eta r)\sinh \eta r\}. \tag{16}$$

Transforming back to the original dependent variables we find

$$K(r) = \frac{2p\eta r\, e^{\eta r}}{p^2 e^{2\eta r} - 1} \tag{17a}$$

$$H(r) = -\eta r\left(\frac{p^2 e^{2\eta r} + 1}{p^2 e^{2\eta r} - 1}\right) + 1 \tag{17b}$$

where $p = a/[b - (b^2 + a^2)^{1/2}]$ is an arbitrary constant. This coincides with the general point monopole solution obtained by Ju (1978). The regular PS solution

$$K(r) = \frac{\eta r}{\sinh \eta r} \qquad H(r) = -\eta r \coth \eta r + 1 \tag{18}$$

can be derived from (17) by setting $p = 1$. By putting $p = e^{\alpha}$ we get the solution reported by Protogenov (1977):

$$K(r) = \frac{\eta r}{\sinh(\eta r + \alpha)} \qquad H(r) = -\eta r \coth(\eta r + \alpha) + 1. \tag{19}$$

We were not able to sum $B(r)$ and $C(r)$ series for non-zero a_1, b_1, c_1, $d_1 = e_1$ with all other constants set to zero. However, the $A(r)$ series can be summed in this case because it possesses a representative term. The function $A(r)$ obtained after summation can be substituted in (8a) to obtain an uncoupled non-linear equation in B. This equation can be further reduced to the one-dimensional Liouville equation and, from the known solutions of it, $B(r)$ can be obtained. From the knowledge of $A(r)$ and $B(r)$, $K(r)$ can be evaluated. $H(r)$ can be constructed by direct substitution of $K(r)$ in (1a). However, this procedure does not give any new result. Details of these calculations are discussed in the appendix.

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Appendix

The $(n+1)$th term of the $A(r)$ series is given by

$$A = ar(-1)^n d^n \sum_{k=0}^{n-1} (-1)^{k+1} \frac{(cr/d)^{k+1}}{(k+1)!} \binom{n-1}{k} \tag{A1}$$

where we have relabelled a_1, c_1 and d_1 as a, c and d respectively. From this

$$\begin{aligned} A_{n+2} &= ar(-1)^n d^{n+1} \sum_{k=0}^{n} (-1)^k \frac{(cr/d)^{k+1}}{(k+1)!} \frac{n!}{k!(n-k)!} \\ &= ar(-1)^n \frac{d^{n+1}}{n+1} \left(\frac{cr}{d}\right) \sum_{k=0}^{n} (-1)^k \frac{(cr/d)^k}{k!} \binom{n+1}{k+1} \\ &= ar(-1)^n \frac{d^{n+1}}{n+1} \left(\frac{cr}{d}\right) \sum_{k=0}^{n} (-1)^k \frac{(cr/d)^k}{k!} \binom{n+1}{n-k} \\ &= ar(-1)^n \frac{d^{n+1}}{n+1} \frac{cr}{d} L_n^1(cr/d) \end{aligned} \tag{A2}$$

where $L_n^1(x)$ is the associated Laguerre polynomial (Gradshteyn and Ryzhik 1965). $A(r)$ now becomes

$$A(r) = \varepsilon ar \left[1 + \left(\frac{cr}{d}\right) \sum_{n=0}^{\infty} (-1)^n \frac{(\varepsilon d)^{n+1}}{n+1} L_n^1\left(\frac{cr}{d}\right) \right]. \tag{A3}$$

Using the relation (Whittaker and Watson 1965)

$$\sum_{n=0}^{\infty} a_n d^n = \int_0^{\infty} \mathrm{e}^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n d^n}{n!} \, \mathrm{d}t \tag{A4}$$

$A(r)$ can be rewritten as

$$A(r) = ar\left(1 + cr \int_0^{\infty} \mathrm{d}t \, \mathrm{e}^{-t} \sum_{n=0}^{\infty} (-1)^n \frac{(td)^n L_n^1(cr/d)}{(n+1)!} \right). \tag{A5}$$

After performing the summation (Gradshteyn and Ryzhik 1965) we get

$$A(r) = \varepsilon ar\left(1 + \varepsilon cr \int_0^{\infty} \mathrm{d}t (-crt\varepsilon)^{-1/2} \, \mathrm{e}^{-(1+d\varepsilon)t} J_1[2(-crt\varepsilon)^{1/2}] \right) \tag{A6}$$

which upon integration (Gradshteyn and Ryzhik 1965) yields, for $c<0$,

$$A(r) = \varepsilon ar \exp\left(\frac{\varepsilon cr}{1+d\varepsilon} \right). \tag{A7}$$

Substituting this in $(8a)$, an uncoupled non-linear equation for B is obtained:

$$D^2 BB = -2(\varepsilon a)^2 \exp[2\varepsilon cr/(1+d\varepsilon)]. \tag{A8}$$

By putting

$$B(r)=\varepsilon a\,\exp\left(\frac{\varepsilon cr}{1+d\varepsilon}-f(r)\right) \tag{A9}$$

we get the one-dimensional Liouville equation

$$f''=e^{2f}. \tag{A10}$$

Three distinct solutions of this equation are (Ju 1978)

$$f=-\ln(r+\beta) \tag{A11a}$$

$$f=-\ln\left(\frac{\sin[\alpha(r+\beta)]}{\alpha}\right) \tag{A11b}$$

$$f=-\ln\left(\frac{\sinh[\alpha(r+\beta)]}{\alpha}\right) \tag{A11c}$$

where α and β are arbitrary constants. Using (A7), (A9) and (3), $K(r)$ can be calculated for each solution (A11). In each case $H(r)$ can be constructed by direct substitution of $K(r)$ in (1a). The results are, for (A11a),

$$K=r/(r+\beta) \qquad H=\beta/(\beta+r) \tag{A12}$$

for (30b)

$$K=\frac{\alpha r}{\sin\alpha(r+\beta)} \qquad H=-2\alpha r\cot[\alpha(r+\beta)]+1 \tag{A13}$$

and for (30c)

$$K=\frac{2\alpha r}{\sinh[\alpha(r+\beta)]} \qquad H=-\alpha r\coth[\alpha(r+\beta)]+1. \tag{A14}$$

The solutions (A12), (A13) and (A14) are readily identified as solutions (13), (19) and (17) respectively with a simple redefinition of parameters.

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